MATHEMATICAL TRIPOS **Dr J.P. Dougherty**

ELECTROMAGNETISM (O5) Michaelmas 1998

VECTOR CALCULUS: A summary of useful results with some examples

1. Vector Algebra in $\mathbb{R}^3$

Addition, multiplication by scalars, components, unit orthogonal vectors as base (right hand triad convention). Scalar and vector products:

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{1}$$

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) \tag{2}$$

Distributive laws.

Suffix notation

a_i represents $\mathbf{a}$. Definitions and use of δ_{ij} and ϵ_{ijk}. The identity

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{3}$$

Scalar and Vector Triple products.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c} \tag{4}$$

Invariance of these constructions under rotation of axes.

2. Scalar and vector fields

Idea of $\phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$.

First derivatives

$$\text{grad}\phi = \nabla\phi \text{ and has } i\text{-component } \frac{\partial\phi}{\partial x_i} \tag{5}$$

$$\text{div}\mathbf{A} = \nabla.\mathbf{A} = \delta_{ij}\frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial x_i} \tag{6}$$

$$\text{curl}\mathbf{A} = \nabla \times \mathbf{A} \text{ and has } i\text{-component } \epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \tag{7}$$

Second derivatives

$$\text{div grad}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x_i\partial x_i} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} \tag{8}$$

$$\text{curl grad}\phi \equiv 0 \tag{9}$$

$$\text{div curl}\mathbf{A} \equiv 0 \tag{10}$$

$$\text{curl curl}\mathbf{A} \equiv \text{grad div}\mathbf{A} - \nabla^2\mathbf{A} \tag{11}$$

Here $\nabla^2\mathbf{A}$ means that the operator ∇^2 as above acts on each cartesian component of $\mathbf{A}$ in turn. The situation is more complicated in curvilinear coordinates (see below).

Existence of scalar and vector potentials

We work with differentiable vector fields in a simply-connected volume V.

(i) $\text{curl}\mathbf{E} = 0$ in $V \Leftrightarrow \exists\ \phi$ such that $\mathbf{E} = -\nabla\phi$ (12)

(ii) $\text{div}\mathbf{B} = 0$ in $V \Leftrightarrow \exists\ \mathbf{A}$ such that $\mathbf{B} = \text{curl}\mathbf{A}$ (13)

In case (i) the $-$ sign is conventional in this subject, and ϕ is fixed except for an additive constant. More will be said about case (ii) in lectures.

Derivatives of products

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{14}$$

$$\text{div}(\phi\mathbf{A}) = \phi\text{div}\mathbf{A} + \mathbf{A}.\nabla\phi \tag{15}$$

$$\text{curl}(\phi\mathbf{A}) = \phi\text{curl}\mathbf{A} + \nabla\phi \times \mathbf{A} \tag{16}$$

$$\text{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.\text{curl}\mathbf{A} - \mathbf{A}.\text{curl}\mathbf{B} \tag{17}$$

$$\nabla(\mathbf{A}.\mathbf{B}) = (\mathbf{A}.\nabla)\mathbf{B} + (\mathbf{B}.\nabla)\mathbf{A} + \mathbf{A}\times\text{curl}\mathbf{B} + \mathbf{B}\times\text{curl}\mathbf{A} \tag{18}$$

$$\text{curl}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\text{div}\mathbf{B} - \mathbf{B}\text{div}\mathbf{A} + (\mathbf{B}.\nabla)\mathbf{A} - (\mathbf{A}.\nabla)\mathbf{B} \tag{19}$$

3. Integral Theorems

Green's Theorem in $\mathbb{R}^2$

Denote the vector field by $(u(x,y), v(x,y))$. Let $S \subset \mathbb{R}^2$ be a simply connected domain, and C its boundary curve taken anti-clockwise. Then

$$\int_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy = \oint_C (udy - vdx) \tag{20}$$

[Note this is the prototype of both the divergence and Stokes theorems below.]

Divergence Theorem in $\mathbb{R}^3$

$\mathbf{F}(\mathbf{r})$ is a vector field defined in $V \subset \mathbb{R}^3$; V has surface S , and $d\mathbf{S}$ denotes a surface element parallel to the outgoing normal.

$$\int_V \text{div}\mathbf{F}dV = \int_S \mathbf{F}.d\mathbf{S} \tag{21}$$

Corollaries:

$$\int_V \nabla\phi dV = \int_S \phi d\mathbf{S} \tag{22}$$

(put $\mathbf{F} = \mathbf{k}\phi$, $\mathbf{k}=$ constant).

$$\int_V \text{curl}\mathbf{A}dV = \int_S d\mathbf{S} \times \mathbf{A} \tag{23}$$

(put $\mathbf{F} = \mathbf{A} \times \mathbf{k}$).

$$\int_V \left(\phi\nabla^2\phi + (\nabla\phi)^2\right) dV = \int_S \phi\nabla\phi.d\mathbf{S} \tag{24}$$

(put $\mathbf{F} = \phi\nabla\phi$).

Green's Reciprocal Theorem:

$$\int_V (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1)dV = \int_S (\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1).d\mathbf{S} \tag{25}$$

Stokes's Theorem in $\mathbb{R}^3$

S is an open orientable surface (no Möbius bands!), bounded by a closed curve C, and lies entirely within a simply connected volume within which $\mathbf{F}(\mathbf{r})$ is defined and differentiable. The sign of the normal to S and the sign of positive progress around C are related as exemplified by Green's theorem in $\mathbb{R}^2$ as above. Then

$$\int_S \text{curl}\mathbf{F}.d\mathbf{S} = \oint_C \mathbf{F}.\mathbf{dr} \tag{26}$$

Corollaries:

$$\int_S \mathbf{dS} \times \nabla\phi = \oint_C \phi d\mathbf{r} \tag{27}$$

See also example 7 below.

4. Derivatives of r

Here $r \neq 0$ throughout.

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = \text{unit vector along } \mathbf{r} \text{ (as is geometrically obvious!)} . \tag{28}$$

$$\frac{\partial r^n}{\partial x_i} = nr^{n-2}x_i \tag{29}$$

In particular:

$$\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{x_i}{r^3} \tag{30}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} \tag{31}$$

$$\nabla^2\left(\frac{1}{r}\right) = 0 \tag{32}$$

Given two points $\mathbf{r}, \mathbf{r}'$ with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$

$$\frac{\partial}{\partial x_i}\frac{1}{R} = -\frac{\partial}{\partial x_i'}\frac{1}{R} = -\frac{x_i - x_i'}{R^3} \tag{33}$$

5. Solid angle

Given a surface element $d\mathbf{S}$ not at 0, the *solid angle* that it subtends at 0 is defined by $d\Omega = \mathbf{r}.d\mathbf{S}/r^3$. This is the area of the radial projection of $d\mathbf{S}$ onto the unit sphere centred at 0.

For an orientable S, bounded by C, the total solid angle subtended at 0 is

$$\Omega = \int_S \frac{\mathbf{r}.d\mathbf{S}}{r^3} \tag{34}$$

If S is a closed surface

$$\Omega = \begin{cases} 4\pi & \text{if 0 is inside } S, \\ 0 & \text{if 0 is outside } S \end{cases} \tag{35}$$

6. Expressions for grad, div, curl **and ∇^2 in curvilinear coordinates**

(Note that where items are abbreviated as "..." these should be filled in by analogy with the terms given, using obvious permutations of $\{1, 2, 3\}$.)

General

Write coordinates as $(\lambda_1, \lambda_2, \lambda_3)$ and define $\{h_i\}$ by $d\mathbf{r} = (h_1 d\lambda_1, h_2 d\lambda_2, h_3 d\lambda_3)$

$$\nabla\psi = \left(\frac{1}{h_1}\frac{\partial\psi}{\partial\lambda_1}, ..., ...\right) \tag{36}$$

$$\text{div}\mathbf{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial\lambda_1}(h_2 h_3 A_1) + ... + ...\right) \tag{37}$$

$$\text{curl}\mathbf{A} = \left(\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial\lambda_2}(h_3 A_3) - \frac{\partial}{\partial\lambda_3}(h_2 A_2)\right), ..., ...\right) \tag{38}$$

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial\lambda_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial\lambda_1}\right) + ... + ...\right) \tag{39}$$

Cylindrical polars

Coordinates (r, θ, z). Note $\lambda_1 = 1, \lambda_2 = r, \lambda_3 = 1$.

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial z}\right) \tag{40}$$

$$\mathrm{div}\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \tag{41}$$

$$\mathrm{curl}\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right) \tag{42}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} \tag{43}$$

Spherical polars

Coordinates (r, θ, ϕ). Note $\lambda_1 = 1, \lambda_2 = r, \lambda_3 = r\sin\theta$.

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right) \tag{44}$$

$$\mathrm{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{45}$$

$$\begin{aligned}\mathrm{curl}\mathbf{A} = \Bigg(&\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right), \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right),\\ &\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\Bigg)\end{aligned} \tag{46}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{47}$$

7. Force systems

Consider a set of forces $\{\mathbf{F}_i\}$ which act at points $\{\mathbf{r}_i\}$. Define the *total force* by

$$\mathbf{F} = \sum_i \mathbf{F}_i \tag{48}$$

The *total moment* about a point $\mathbf{a}$ is defined by

$$\mathbf{G}(\mathbf{a}) = \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i \tag{49}$$

Note

$$\mathbf{G}(\mathbf{a}) = \mathbf{G}(\mathbf{0}) - \mathbf{a} \times \mathbf{F} \tag{50}$$

A system is *null* if $\mathbf{F}=\mathbf{G}=\mathbf{0}$. Two systems are *equivalent* if they have the same values of $\mathbf{F}$ and $\mathbf{G}$. If $\mathbf{F} = \mathbf{0}$, $\mathbf{G}$ is independent of $\mathbf{a}$, and the system is a *couple*.

REVISION EXAMPLES

These are intended to restore fluency and confidence; do some, all or none of questions 1-7, as you feel appropriate. It is not intended that this should occupy a whole week's supervision time.

1. Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

2. Force Systems. If $\mathbf{F} \neq \mathbf{0}$, $\mathbf{F}.\mathbf{G}(\mathbf{a})$ is invariant to the choice of $\mathbf{a}$ and it is possible to choose $\mathbf{a}$ such that $\mathbf{F}$ and $\mathbf{G}$ are parallel. Verify this by taking

$$\mathbf{a} = \frac{\mathbf{F} \times \mathbf{G}(\mathbf{0})}{\mathbf{F^2}} + k\mathbf{F}$$

where k is arbitrary. The system is then equivalent to a single force $\mathbf{F}$ acting at $\mathbf{a}$ together with a couple parallel to $\mathbf{F}$ (cf. corkscrew!).

3. If $\mathbf{B} = (0, 0, B_0)$ in cartesians, verify that the following possible vector potentials yield $\mathbf{B} = \mathrm{curl}\mathbf{A}$:

(i) In cartesians, $\mathbf{A} = (0, xB_0, 0)$

(ii) In cylindrical polars, $\mathbf{A} = (0, \frac{1}{2}B_0 r, 0)$

(iii) In spherical polars, $\mathbf{A} = (0, 0, \frac{1}{2}B_0 r \sin\theta)$

4. If $\mathbf{B} = \mathrm{curl}\mathbf{A}$ where (for $r \neq 0$), the vector potential is $\mathbf{A} = \mathbf{M} \times \mathbf{r}/r^3$, $\mathbf{M} = \text{constant}$, show that $\mathbf{B}$ can also be written as $\mathbf{B} = -\nabla\chi$, where the scalar potential is $\chi = \mathbf{M}.\mathbf{r}/r^3$.

[*Hints*: relate $\mathbf{r}/r^3$ to $\nabla(1/r)$ and work throughout in cartesian suffix notation; recall that $\nabla^2(1/r) = 0$].

5. Show that:

(i) $\int_S r^n \mathbf{r}.d\mathbf{S} = \int_V (n+3) r^n dV$, (ii) $\int_S r^n d\mathbf{S} = \int_V n r^{n-2} \mathbf{r} dV$

6. Show that the solid angle of a cone of semi-vertical angle α is $2\pi(1 - \cos\alpha)$. What is the solid angle subtended by one face of a cube at its centre ?

7. Establish a second corollary to Stokes's theorem:

$$\int_S \Big((-\mathrm{div}\mathbf{A}) d\mathbf{S} + \nabla(\mathbf{A}.d\mathbf{S}) \Big) = \oint_C d\mathbf{r} \times \mathbf{A}$$

[Here the "∇" acts only on $\mathbf{A}$].

8. *This is very hard and long, aimed at enthusiasts or budding Lucasian Professors. Do* **not** *plague your supervisors with it!*

Prove Lundgren's identity:

$$\begin{aligned} \nabla\, [\mathbf{A}, \mathbf{B}, \mathbf{C}] = \mathbf{A} \times \mathrm{curl}(\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times \mathrm{curl}(\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times \mathrm{curl}(\mathbf{A} \times \mathbf{B}) \\ -(\mathbf{B} \times \mathbf{C})\ \mathrm{div}\mathbf{A} - (\mathbf{C} \times \mathbf{A})\ \mathrm{div}\mathbf{B} - (\mathbf{A} \times \mathbf{B})\ \mathrm{div}\mathbf{C} \end{aligned}$$

(Here [...] denotes the scalar triple product).

[Reference: T.S. Lundgren, *Physics of Fluids* **6**, 898 (1963)].

Comments on both the Notes and Examples welcomed! email: jpd2@damtp.cam.ac.uk